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Row Convergence Theorems for Vector-Valued Padé Approximants*

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Yet another method of proof of de Montessus' 1902 theorem is given. We show how this proof readily extends to row convergence theorems for four different kinds of vector Padé approximants. These approximants all belong to the category associated with vector-valued C-fractions formed using generalised inverses. The proof of a conjecture by Graves-Morris and Saff (*J. Comput. Appl. Math.* **23**, 1988, 63–85) is given and new row convergence theorems for hybrid vector Padé approximants are proved. © 1997 Academic Press

1. INTRODUCTION

de Montessus' 1902 theorem [16] is the prototype row convergence theorem for Padé approximants. Under the principal hypothesis that $f(z)$ is a function which is analytic at the origin, meromorphic in a disk $|z| < r$, and having precisely m poles in that disk, de Montessus essentially showed that the row sequence of Padé approximants of type $[l/m]$ converges to $f(z)$ as $l \to \infty$ in the disk, except near the poles of $f(z)$. His theorem has numerous proofs which facilitate different generalisations. In this paper we

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introduce yet another proof of de Montessus' theorem based on a Cauchy representation of the MacLaurin coefficients of $f(z)$. More importantly, we are concerned with extensions of de Montessus' theorem to the vector case when the function $\mathbf{f}(z)$ has MacLaurin coefficients $\mathbf{c}_i \in \mathbb{C}^d$.

Convergence theorems for vector Padé approximants usually fall into two categories. One category is alternatively called simultaneous Padé approximants [1, 2]. Analogues of de Montessus' theorem for these approximants have been established [11, 24]. In this paper, we are concerned with the other category, in which vector inverses are directly or indirectly involved. The first vector-valued corresponding continued fraction was introduced by Wynn [26, 27]. Typically, these fractions take the form

$$\mathbf{F}(z) = \mathbf{b}_0 + \frac{z}{\mathbf{b}_1 +} \frac{z}{\mathbf{b}_2 +} \frac{z}{\mathbf{b}_3 +} \cdots \tag{1.1}$$

with $\mathbf{b}_i \in \mathbb{C}^d$ and $z \in \mathbb{C}$. The convergents of these fractions are interpreted and evaluated using Moore–Penrose generalised inverses. For a vector $\mathbf{v} := (v_1, v_2, ..., v_d) \in \mathbb{C}^d$, this inverse is defined by

$$\mathbf{v}^{-1} := \mathbf{v}^*/|\mathbf{v}|^2,$$

where the asterisk denotes the complex conjugate, and $|\mathbf{v}|^2 := \sum_{i=1}^{d} v_i v_i^*$. The associated vector Padé approximants typically take the form

$$\mathbf{R}(z) = \mathbf{P}(z)/Q(z), \tag{1.2}$$

where $\mathbf{P}(z)$ is a vector numerator polynomial of degree n at most and $Q(z)$ is a scalar denominator polynomial of even degree $2k$ precisely [5]. The approximants of types $[2k/2k]$ and $[2k+1/2k]$ would correspond to the convergents of (1.1). By taking $z = 1$, the approximants are related to entries in the vector epsilon table by

$$\mathbf{R}(1) = \varepsilon_{2k}^{(n-2k)}, \qquad n = 2k, 2k+1, ... \tag{1.3}$$

In general terms, McLeod [15] showed that the entries $\varepsilon_{2k}^{(j)}$ of (1.3), for fixed k, sum a k-component vector-valued geometric series. Graves-Morris and Saff [12] proved the first row convergence theorem for the convergence of vector Padé approximants of types $[n/2k]$ with k fixed for the MacLaurin series of a vector-valued meromorphic function having precisely k poles in a disk $|z| < r$, much as in de Montessus' 1902 theorem. An important consequence of this result is that it proved, in general terms, that the column $\varepsilon_{2k}^{(j)}$ of the vector epsilon table converges rapidly as $j \to \infty$ to the sum of a k-component vector-valued geometric series. Because vector Padé approximants which converge to a vector-valued meromorphic function $\mathbf{f}(z)$ having k poles in a disk have denominators of degree $2k$ and not k, hybrid vector Padé approximants of type $[n/k]$ were introduced by Graves-Morris [7]. These approximants also satisfy a McLeod-type theorem [8], and Roberts has shown that they have a number of attractive properties [22].

In Section 2, we introduce new integral representations of the denominator polynomial of various types of vector Padé approximants based on a Cauchy representation of the coefficients of the given MacLaurin series. Once the denominator polynomials have been specified, the actual approximants are constructed using the accuracy-through-order principle, as are Padé type approximants.

In Section 3, we give a proof of a row convergence theorem which applies equally well to vector-valued Padé approximants and to their hybrid form. The method of proof applies likewise to the approximants derived in the framework of a complex Clifford algebra [18]. In fact, we obtain four distinct convergence theorems, each analogous to de Montessus' theorem. An important conclusion is that rapid convergence is established for several kinds of vector-valued Padé approximants for vector series which are dominated by precisely k geometric components. As a bonus, the conjecture of Graves-Morris and Saff [12] is, in large measure, proved.

2. NEW FORMS FOR THE DENOMINATOR POLYNOMIALS

In this section, we introduce new integral representations of the denominator polynomials of several different types of vector Padé approximants. Except in degenerate cases [10], these polynomials are unique up to an unimportant constant multiplier.

In Subsection 2.1, we derive formulae for the denominator polynomial $\phi(z)$ of degree $2k$ of a vector Padé approximant of type $[n/2k]$ associated with vector-valued continued fractions which are interpreted (and can be evaluated) using generalised inverses. In Subsection 2.2, we state the equivalent formula for the denominator polynomial $\sigma(z)$ of degree k for a hybrid vector Padé approximant of type $[n/k]$.

In Subsection 2.3, we state the formula for a denominator polynomial $\phi^{\mathrm{C}}(z)$ of degree $2k$ of a vector Padé approximant of type $[n/2k]$ associated with a vector-valued continued fraction formulation which is interpreted using a complex Clifford algebra. We also state a similar formula for the denominator polynomial $\sigma^{\mathrm{C}}(z)$, of degree k, of the associated hybrid approximant.

2.1. *The Denominator Polynomial of a Generalised Inverse, Vector-Valued Padé Approximant*

The determinantal formula for the denominator polynomial of a generalised inverse vector-valued Padé approximant of type $[n/2k]$ for the function

$$\mathbf{f}(z) = \mathbf{c}_0 + \mathbf{c}_1 z + \mathbf{c}_2 z^2 + ..., \qquad \mathbf{c}_i \in \mathbb{C}^d \tag{2.1}$$

is

$$Q(z) = \begin{vmatrix} 0 & M_{12} & \cdots & M_{1,2k} & M_{1,2k+1} \\ M_{21} & 0 & \cdots & M_{2,2k} & M_{2,2k+1} \\ \vdots & \vdots & & \vdots & \vdots \\ M_{2k,1} & M_{2k,2} & \cdots & 0 & M_{2k,2k+1} \\ z^{2k} & z^{2k-1} & \cdots & z & 1 \end{vmatrix}, \tag{2.2}$$

where

$$\begin{aligned} M_{ij} &:= \sum_{l=0}^{j-i-1} \mathbf{c}_{l+i+n-2k} \cdot \mathbf{c}^*_{j-l+n-2k-1} && \text{if } \ i<j \\ &:= -M_{ji} && \text{if } \ j \leqslant i \end{aligned} \tag{2.3}$$

for $1 \leqslant i, j \leqslant 2k+1$. Equation (2.3) also defines the matrix $M \in \mathbb{R}^{(2k+1)\times(2k+1)}$. From (2.1) we find that

$$\mathbf{c}_j = \frac{1}{2\pi \mathrm{i}} \int_C x^{-j-1} \mathbf{f}(x)\, dx, \tag{2.4}$$

where C is the usual Cauchy contour enclosing $x=0$; here $\mathrm{i} = \sqrt{-1}$, and it is to be distinguished from i which we use as an index. We use a dot to denote the scalar product $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{d} a_i b_i$ for $\mathbf{a}, \mathbf{b} \in \mathbb{C}^d$. It is convenient to define

$$\mathbf{e}(x) := \frac{1}{2\pi \mathrm{i}}\, x^{-n-1} \mathbf{f}(x) \tag{2.5}$$

for use in Cauchy integrals.

The functional complex conjugate of a function $f(z)$ having a Taylor expansion

$$f(z) = \sum_{i=0}^{\infty} f_i (z - x_0)^i$$

about any point x_0 on the real axis is

$$f^*(z) := \sum_{i=0}^{\infty} f_i^*(z-x_0)^i,$$

and thence by analytic continuation beyond the circle of convergence. If $f(z)=f^*(z)$, so that the Taylor coefficients of $f(z)$ are real, $f(z)$ is said to be a real symmetric function [23]. We also use a star product, which is a symmetrised scalar product, defined by

$$\mathbf{e}(x) * \mathbf{e}(y) := \tfrac{1}{2}(\mathbf{e}(x)\cdot\mathbf{e}^*(y) + \mathbf{e}^*(x)\cdot\mathbf{e}(y)). \tag{2.6}$$

By adapting the approach of Woodcock and Graves-Morris [25] (who use the umbral calculus), we obtain

$$\begin{aligned}
M_{ij} &= \left(\frac{1}{2\pi\mathrm{i}}\right)^2 \int_{C_2}\int_{C_1} \sum_{l=0}^{j-i-1} x^{-l-i-n+2k-1} y^{-j+l-n+2k}\mathbf{f}(x)\cdot\mathbf{f}^*(y)\,dx\,dy \\
&= \int_{C_2}\int_{C_1} (xy)^{2k+1}\,\frac{x^{-i}y^{-j}-x^{-j}y^{-i}}{x-y}\,\mathbf{e}(x)\cdot\mathbf{e}^*(y)\,dx\,dy \\
&= \int_{C_2}\int_{C_1} (xy)^{2k+1}\,\frac{x^{-i}y^{-j}-x^{-j}y^{-i}}{x-y}\,\mathbf{e}(x)*\mathbf{e}(y)\,dx\,dy \\
&= \left\{\int_{C_2}\int_{C_1} + \int_{C_1}\int_{C_2}\right\} (xy)^{2k+1}\,\frac{x^{-i}y^{-j}}{x-y}\,\mathbf{e}(x)*\mathbf{e}(y)\,dx\,dy \\
&= 2\int_{C}\int_{C} (xy)^{2k+1}\,\frac{x^{-i}y^{-j}}{x-y}\,\mathbf{e}(x)*\mathbf{e}(y)\,dx\,dy,
\end{aligned} \tag{2.7}$$

where the contours C_1, C_2 are taken to lie inside, outside C, respectively, and then the limit $C_1, C_2 \to C$ is taken, so that the x-integral over C in (2.7) is to be understood as a principal value integral. Hence

$$Q(z) = 2^{2k}\int_C\int_C\cdots\int_C V(\mathbf{x},\mathbf{y},z)\prod_{i=1}^{2k}\frac{\mathbf{e}(x_i)*\mathbf{e}(y_i)}{x_i-y_i}\,dx_i\,dy_i, \tag{2.8}$$

where $\mathbf{x} := (x_1, x_2, ..., x_{2k})$, $\mathbf{y} := (y_1, y_2, ..., y_{2k}) \in \mathbb{C}^{2k}$, and

$$V(\mathbf{x},\mathbf{y},z) := \begin{vmatrix} x_1^{2k}y_1^{2k} & \cdots & x_1^{2k}y_1^0 \\ \vdots & & \vdots \\ x_{2k}^1 y_{2k}^{2k} & \cdots & x_{2k}^1 y_{2k}^0 \\ z^{2k} & \cdots & 1 \end{vmatrix}. \tag{2.9}$$

Using the Vandermonde expansion of (2.9) in (2.8), we obtain

$$Q(z)=2^{2k}\int_C\int_C\cdots\int_C\prod_{i<j}^{2k}(y_i-y_j)\prod_{i=1}^{2k}(y_i-z)x_i^{2k+1-i}\frac{\mathbf{e}(x_i)*\mathbf{e}(y_i)}{x_i-y_i}\,dx_i\,dy_i. \tag{2.10}$$

By permuting the pairs (x_i, y_i), $1\leqslant i\leqslant 2k$, in the integrand of (2.10), we obtain

$$Q(z)=\frac{2^{2k}}{(2k)!}\int_C\int_C\cdots\int_C\prod_{i<j}^{2k}(y_i-y_j)(x_i-x_j)$$
$$\times\prod_{i=1}^{2k}\frac{x_i(y_i-z)}{x_i-y_i}\,\mathbf{e}(x_i)*\mathbf{e}(y_i)\,dx_i\,dy_i. \tag{2.11}$$

This formula is our first (principal value) integral representation for $Q(z)$.

An alternative to (2.11) is obtained by applying Cayley's 1857 theorem [3] to the bordered anti-symmetric determinant. To express the theorem, we introduce the notation $X_{\backslash i;j}$ to denote the matrix formed by deletion of row i and column j of a given matrix X, and deletion of extra rows and columns is denoted similarly. We introduce the auxiliary anti-symmetric determinant

$$Z:=\begin{vmatrix} 0 & M_{12} & \cdots & M_{1,2k+1} & z^{2k} \\ M_{21} & 0 & \cdots & M_{2,2k+1} & z^{2k-1} \\ \vdots & \vdots & & \vdots & \vdots \\ M_{2k+1,1} & M_{2k+1,2} & \cdots & 0 & 1 \\ -z^{2k} & -z^{2k-1} & \cdots & -1 & 0 \end{vmatrix}. \tag{2.12}$$

From Cayley's theorem, we have

$$Q(z)=\mathrm{Pf}(Z)\times\mathrm{Pf}(Z_{\backslash 2k+1,2k+2;\,2k+1,2k+2}). \tag{2.13}$$

For the present purposes, the Pfaffian of an anti-symmetric matrix $A\in\mathbb{C}^{2k\times 2k}$ is conveniently defined as

$$\mathrm{Pf}(A):=\frac{1}{2^k k!}\sum_{\sigma}\operatorname{sign}\sigma\prod_{i=1}^{k}A_{\sigma(2i-1),\sigma(2i)} \tag{2.14}$$

[4, 14], where each permutation σ can be expressed as

$$\sigma=\begin{bmatrix} 1 & 2 & \cdots & 2k \\ i_1 & i_2 & \cdots & i_{2k}\end{bmatrix}. \tag{2.15}$$

The factorisation (2.13) involves the polynomial Pfaffian $\mathrm{Pf}(Z)$ multiplied by a constant Pfaffian. To get their integral representations, it is easiest to go back to (2.2), from which we have

$$Q(z)=\sum_{j=0}^{2k} z^{2k-j}(-1)^j \det M_{\backslash 2k+1;\, j+1}\,. \tag{2.16}$$

We apply Cayley's theorem to the (even order, bordered anti-symmetric) determinant of $M_{\backslash j+1;\, 2k+1}$ in (2.16) and deduce that

$$Q(z)=\phi(z)\ \mathrm{Pf}(M_{\backslash 2k+1;\, 2k+1}), \tag{2.17}$$

where

$$\phi(z)=\sum_{j=0}^{2k} z^{2k-j}(-1)^j\ \mathrm{Pf}(M_{\backslash j+1;\, j+1}). \tag{2.18}$$

Hence, using Pfaffian notation [17], we have

$$\phi(0)=\mathrm{Pf}(M_{\backslash 2k+1;\, 2k+1})=\left|\begin{array}{llll} M_{12} & M_{13} & \cdots & M_{1,\,2k} \\ & M_{23} & \cdots & M_{2,\,2k} \\ & & & \vdots \\ & & & M_{2k-1,\,2k} \end{array}\right|. \tag{2.19}$$

From (2.12), (2.13), and (2.17)

$$\phi(z)=\mathrm{Pf}(Z)=\left|\begin{array}{lllll} M_{12} & M_{13} & \cdots & M_{1,\,2k+1} & z^{2k} \\ & M_{23} & \cdots & M_{2,\,2k+1} & z^{2k-1} \\ & & & \vdots & \vdots \\ & & & M_{2k,\,2k+1} & z \\ & & & & 1 \end{array}\right|. \tag{2.20}$$

We obtain an integral representation for $\phi(0)$, given by (2.19), by using (2.7) and (2.14):

$$\phi(0)=\frac{1}{k!}\sum_{\sigma}\operatorname{sign}\sigma\int_C\int_C\cdots\int_C \frac{x_1^{-i_1}y_1^{-i_2}}{x_1-y_1}\cdots\frac{x_k^{-i_{2k-1}}y_k^{-i_{2k}}}{x_k-y_k}$$
$$\times\prod_{i=1}^{k}(x_i\,y_i)^{2k+1}\,\mathbf{e}(x_i)*\mathbf{e}(y_i)\,dx_i\,dy_i\,. \tag{2.21}$$

Similarly, by using (2.7) and (2.14) in (2.18), we obtain

$$\phi(z)=\frac{1}{k!}\int_C\int_C\cdots\int_C D(\mathbf{x},\mathbf{y},z)\prod_{i=1}^{k}\frac{\mathbf{e}(x_i)*\mathbf{e}(y_i)}{x_i-y_i}\,dx_i\,dy_i, \tag{2.22}$$

where

$$D(\mathbf{x},\mathbf{y},z)=\begin{vmatrix} x_1^{2k} & x_1^{2k-1} & \cdots & x_1 & 1\\ y_1^{2k} & y_1^{2k-1} & \cdots & y_1 & 1\\ \vdots & \vdots & & \vdots & \vdots\\ x_k^{2k} & x_k^{2k-1} & & x_k & 1\\ y_k^{2k} & y_k^{2k-1} & \cdots & y_k & 1\\ z^{2k} & z^{2k-1} & \cdots & z & 1 \end{vmatrix}.$$

By expansion of $D(\mathbf{x},\mathbf{y},z)$, we obtain

$$\phi(z)=\frac{1}{k!}\int_C\int_C\cdots\int_C \prod_{i<j}(x_i-x_j)(y_i-y_j)(x_i-y_j)(y_i-x_j)$$
$$\times\prod_{j=1}^{k}(z-x_j)(z-y_j)\,\mathbf{e}(x_j)*\mathbf{e}(y_j)\,dx_j\,dy_j. \tag{2.23}$$

By expanding the star product which is defined in (2.6), we obtain the representation

$$\phi(z)=\frac{1}{k!}\int_C\int_C\cdots\int_C \prod_{i<j}(x_i-x_j)(y_i-y_j)(x_i-y_j)(y_i-x_j)$$
$$\times\prod_{j=1}^{k}(z-x_j)(z-y_j)\,\frac{x_j^{-n-1}y_j^{-n-1}}{(2\pi\mathrm{i})^2}\,\mathbf{f}(x_j)\cdot\mathbf{f}(y_j)\,dx_j\,dy_j. \tag{2.24}$$

Equation (2.23) is our second integral representation for the denominator of a vector Padé approximant. It is only superficially similar to (2.11), and these results are connected by

$$Q(z)=\phi(z)\,\phi(0) \tag{2.25}$$

which follows from (2.17) and (2.18).

2.2. *The Denominator Polynomial for Hybrid, Generalised Inverse, Vector Padé Approximants*

Motivated by a formula similar to (2.23) and the fact that $Q(z)$ is proportional to the square of the Padé denominator $q^{[n-k/k]}(z)$ in the scalar $(d=1)$ case, Graves-Morris [8] introduced a polynomial

$\sigma(z) \approx \sqrt{Q(z)}$. His expressions for $Q(z)$ and $\sigma(z)$ involve the umbral calculus; here we need the equivalent expression based on (2.4), which follows immediately from (2.24) and it is

$$\sigma(z) = \frac{1}{k!} \int_C \int_C \cdots \int_C \prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)(x_i - y_j)(y_i - x_j) \times \prod_{j=1}^{k} (z - x_j) \frac{x_j^{-n-1} y_j^{-n-1}}{(2\pi \mathrm{i})^2} \mathbf{f}(x_j) \cdot \mathbf{f}^*(y_j)\, dx_j\, dy_j. \tag{2.26}$$

2.3. *Denominator Polynomials Based on Inverses from* $\mathrm{Cl}(\mathbb{C}^d)$

Complex-valued vectors can be represented by elements of a Clifford algebra in different ways. Roberts [18] gave a different formulation of a vector-valued Padé approximation using a complex (universal) Clifford $\mathrm{Cl}(\mathbb{C}^d)$ in which the inverse of a vector $\mathbf{v} \in \mathbb{C}^d$ is taken to be

$$\mathbf{v}^{-1} := \mathbf{v}(\mathbf{v} \cdot \mathbf{v})^{-1},$$

where $\mathbf{v} \cdot \mathbf{v} = \sum_{j=1}^{d} v_j^2$, provided that $\mathbf{v} \cdot \mathbf{v} \neq 0$.

With this approach, the real symmetric denominator polynomial $Q(z)$ is replaced by $Q^{\mathrm{C}}(z)$, also given by (2.2) but with the complex-valued matrix elements

$$\begin{aligned} M_{ij}^{\mathrm{C}} &:= \sum_{l=0}^{j-i-1} \mathbf{c}_{l+i+n-2k} \cdot \mathbf{c}_{j-l+n-2k-1} && \text{if} \quad i<j \\ &:= -M_{ji}^{\mathrm{C}} && \text{if} \quad j \leqslant i \end{aligned} \tag{2.27}$$

instead of (2.3). From the results (2.24) and (2.26), it follows directly that the corresponding denominator polynomials in $\mathbb{C}[z]$ are

$$\phi^{\mathrm{C}}(z) := \frac{1}{k!} \iint \cdots \int \prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)(x_i - y_j)(y_i - x_j) \times \prod_{j=1}^{k} (z - x_j)(z - y_j) \frac{x_j^{-n-1} y_j^{-n-1}}{(2\pi \mathrm{i})^2} \mathbf{f}(x_j) \cdot \mathbf{f}(y_j)\, dx_j\, dy_j \tag{2.28}$$

which has degree $2k$, and its corresponding hybrid version

$$\sigma^{\mathrm{C}}(z) := \frac{1}{k!} \iint \cdots \int \prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)(x_i - y_j)(y_i - x_j) \times \prod_{j=1}^{k} (z - x_j) \frac{x_j^{-n-1} y_j^{-n-1}}{(2\pi \mathrm{i})^2} \mathbf{f}(x_j) \cdot \mathbf{f}(y_j)\, dx_j\, dy_j, \tag{2.29}$$

which has degree k. Obviously, there is no distinction between (2.3) and (2.27), or (2.22) and (2.28), or (2.26) and (2.29) if $\mathbf{c}_i \in \mathbb{R}^d$.

In Section 3 we prove a row convergence theorem for the hybrid vector Padé approximants whose specification is based on $\sigma(z)$ as defined by (2.26).

3. ROW CONVERGENCE THEOREMS

In Subsection 2.1, we derived (2.23) as an integral representation of the denominator polynomials $\phi(z)$ required for forming vector Padé approximants of type $[n/2k]$. The corresponding numerator polynomials will be expressed using Nuttall's notation, in which

$$[\psi(z)]_0^n := \sum_{i=0}^{n} \psi_i z^i$$

denotes the $n+1$ term MacLaurin section of $\psi(z)$. The ordinary vector Padé approximant is

$$\mathbf{R}(z) = \mathbf{P}(z)/\phi(z), \tag{3.1}$$

where the numerator polynomial is

$$\mathbf{P}(z) := [\mathbf{f}(z)\,\phi(z)]_0^n. \tag{3.2}$$

When the hybrid form is required, the approximant is

$$\boldsymbol{\rho}(z) = \boldsymbol{\pi}(z)/\sigma(z), \tag{3.3}$$

where $\sigma(z)$ is given by (2.26) and the numerator $\boldsymbol{\pi}(z)$ is

$$\boldsymbol{\pi}(z) := [\mathbf{f}(z)\,\sigma(z)]_0^n. \tag{3.4}$$

In the previous subsection, complex-valued denominator polynomials $\phi^{\mathrm{C}}(z)$ and $\sigma^{\mathrm{C}}(z)$ were defined by (2.28) and (2.29) for cases where $\mathbf{c}_i \in \mathbb{C}^d$ (rather than $\mathbb{R}^d$). The corresponding approximants are similarly defined as

$$\mathbf{R}^{\mathrm{C}}(z) = \mathbf{P}^{\mathrm{C}}(z)/\phi^{\mathrm{C}}(z), \qquad \boldsymbol{\rho}^{\mathrm{C}}(z) = \boldsymbol{\pi}^{\mathrm{C}}(z)/\sigma^{\mathrm{C}}(z), \tag{3.5}$$

where

$$\mathbf{P}^{\mathrm{C}}(z) = [\mathbf{f}(z)\,\phi^{\mathrm{C}}(z)]_0^n, \qquad \boldsymbol{\pi}^{\mathrm{C}}(z) = [\mathbf{f}(z)\,\sigma^{\mathrm{C}}(z)]_0^n. \tag{3.6}$$

We state and prove row convergence theorems for vector-valued Padé approximants and hybrid vector-valued Padé approximants formed with both the standard real-valued and then the complex-valued denominator polynomials. To formulate these results, a dot is used to denote the leading coefficient of a polynomial and a hat is used to denote the monic form of a polynomial. Thus

$$\hat{\phi}(z) = \phi(z)/\dot{\phi} \tag{3.7}$$

is the monic polynomial derived from $\phi(z)$. We will consider functions $\mathbf{f}(z)$ having precisely k poles at $z_1, z_2, ..., z_k$ (allowing repetition if $\mathbf{f}(z)$ has a multipole), and we define

$$\omega(z) = \prod_{i=1}^{k} (z - z_i). \tag{3.8}$$

We require $\mathbf{f}(z)$ to be analytic at the origin, and the poles to be ordered so that

$$0 < |z_1| \leqslant |z_2| \leqslant \cdots \leqslant |z_k| < r. \tag{3.9}$$

Thus $\mathbf{f}(z)$ is assumed to be representable by

$$\mathbf{f}(z) = \frac{\mathbf{g}(z)}{\omega(z)}, \tag{3.10}$$

where $\mathbf{g}(z)$ is analytic in $|z| < r$ and $\mathbf{g}(z_i) \neq \mathbf{0}$ for $i = 1, 2, ..., k$. We refer to any case in which two or more of the z_i are equal as a *confluence* of the poles. We also define a punctured open disk by

$$D_r^- := \{z \in \mathbb{C} : |z| < r\} - \bigcup_{i=1}^{k} \{z_i\}. \tag{3.11}$$

We next state a theorem of Graves-Morris and Saff and give a proof of it based on the Pfaffian decomposition (2.13). By using (2.23) as a representation of the denominator polynomial of a vector Padé approximant, cases of confluence are much more easily handled.

THEOREM 3.1. (Graves-Morris and Saff [12]). *A vector-valued function* $\mathbf{f}(z)$ *is given in the form of a MacLaurin series* (2.1) *and also by* (3.10) *in which it is assumed that* $\omega(z)$ *is real symmetric and that*

$$\mathbf{g}(z_i) \cdot \mathbf{g}^*(z_i) \neq 0, \qquad i = 1, 2, ..., k. \tag{3.12}$$

Let $\mathbf{R}_n(z)$ *be the vector-valued Padé approximant of type* $[n/2k]$ *for* $\mathbf{f}(z)$, *as in* (3.1), *and let* $\phi_n(z)$ *be its associated denominator polynomial, as defined by* (2.23). *Then*

$$\lim_{n\to\infty} \mathbf{R}_n(z)=\mathbf{f}(z), \qquad z\in D_r^- \tag{3.13}$$

and the rate of convergence is governed by

$$\limsup_{n\to\infty} \|\mathbf{f}-\mathbf{R}_n\|_K^{1/n} \leqslant \mu/r, \tag{3.14}$$

where K *is any compact subset of* $D_r^- \cap \{z\in\mathbb{C} : |z| \leqslant \mu\}$ *for any* $\mu<r$.

The (monic) denominator polynomials converge as

$$\lim_{n\to\infty} \hat{\phi}_n(z)=(\omega(z))^2 \tag{3.15}$$

and their rate of convergence is governed by

$$\limsup_{n\to\infty} \|\hat{\phi}_n-\omega^2\|_E^{1/n} \leqslant |z_k|/r, \tag{3.16}$$

where E *is any compact subset of* $\mathbb{C}$.

Proof. We substitute (3.12) into (2.23) and use (2.5) to obtain

$$\begin{aligned}\phi_n(z)=\frac{1}{k!}\iint\cdots\int \prod_{i<j}^{k} (x_i-x_j)(y_i-y_j)(x_i-y_j)(y_i-x_j)\\ \times \prod_{j=1}^{k} (z-x_j)(z-y_j)\,\frac{x_j^{-n-1}y_j^{-n-1}}{(2\pi \mathrm{i})^2}\,\frac{\mathbf{g}(x_j)*\mathbf{g}(y_j)}{\omega(x_j)\,\omega(y_j)}\,dx_j\,dy_j,\end{aligned} \tag{3.17}$$

where the contours for each x_j, y_j integration can now be taken to be $|x_j|=\varepsilon$, $|y_j|=\varepsilon$ with $0<\varepsilon<|z_1|$. We expand all these contours to $|z|=r'$ for any r' satisfying $|z_k|<r'<r$. Assuming, for the moment, that the poles of $\mathbf{f}(z)$ are distinct, we use the residue theorem and the fact that $\omega(z)$ is real symmetric to obtain the dominant term of $\phi_n(z)$, for large n. It is defined as

$$\begin{aligned}\varDelta_n(z):=\frac{1}{k!}\sum_{l_1=1}^{k}\cdots\sum_{l_k=1}^{k}\sum_{l_1'=1}^{k}\cdots\sum_{l_k'=1}^{k}\prod_{i<j}(z_{l_i}-z_{l_j})(z_{l_i'}-z_{l_j'})(z_{l_i}-z_{l_j'})(z_{l_i'}-z_{l_j})\\ \times\prod_{j=1}^{k}(z-z_{l_j})(z-z_{l_j'})\,z_{l_j}^{-n-1}z_{l_j'}^{-n-1}\,\frac{\mathbf{g}(z_{l_j})\cdot\mathbf{g}^*(z_{l_j'})}{\omega'(z_{l_j})\,\omega'(z_{l_j'})}.\end{aligned} \tag{3.18}$$

To simplify (3.18), suppose that $l_1, l_2, ..., l_k$ have been assigned distinct values in the range $[1, k]$, and likewise for $l'_1, l'_2, ..., l'_k$. Then we define

$$\chi = \prod_{i<j} (z_{l_i} - z_{l'_j})(z_{l'_i} - z_{l_j}).$$

Note that $\chi = 0$ if $l_i = l'_j$ for $i < j$ and also $\chi = 0$ if $l'_i = l_j$ for $i < j$. By tabulating $l_1, l_2, ..., l_k$ against $l'_1, l'_2, ..., l'_k$ we find that $\chi \neq 0$ only if all $l'_i = l_i$. Hence (3.18) simplifies to

$$\Delta_n(z) = \frac{1}{k!} \sum_{l_1=1}^{k} \cdots \sum_{l_k=1}^{k} \prod_{i<j} (z_{l_i} - z_{l_j})^4 \times \prod_{j=1}^{k} (z - z_{l_j})^2 \, z_{l_j}^{-2n-2} \, \frac{\mathbf{g}(z_{l_j}) \cdot \mathbf{g}^*(z_{l_j})}{\omega'(z_{l_j})^2}. \tag{3.19}$$

The summand in (3.19) is completely symmetric under interchange of the indices $l_1, l_2, ..., l_k$, and hence

$$\begin{aligned} \Delta_n(z) &= \prod_{i<j} (z_i - z_j)^4 \prod_{j=1}^{k} (z - z_j)^2 \, z_j^{-2n-2} \, \frac{\mathbf{g}(z_j) \cdot \mathbf{g}^*(z_j)}{\omega'(z_j)^2} \\ &= \prod_{j=1}^{k} (z - z_j)^2 \, z_j^{-2n-2} \mathbf{g}(z_j) \cdot \mathbf{g}^*(z_j) \end{aligned} \tag{3.20}$$

and its monic form is

$$\hat{\Delta}_n(z) = \prod_{j=1}^{k} (z - z_j)^2 = [\omega(z)]^2. \tag{3.21}$$

By taking the contributions from the contours expanded round $|z| = r'$ in (3.17) into account, a similar reduction yields

$$\phi_n(z_j) = O\left(\left|\frac{z_j}{r'}\right|^{2n}\right) \cdot \prod_{i=1}^{k} z_i^{-2n}, \tag{3.22}$$

$$\dot{\phi}_n = \left[\prod_{j=1}^{k} z_j^{-2n-2} \mathbf{g}(z_j) \cdot \mathbf{g}^*(z_j)\right] \cdot \left(1 + O\left(\left|\frac{z_k}{r'}\right|^{n}\right)\right), \tag{3.23}$$

and

$$\phi_n(z) = \Delta_n(z) \left(1 + O\left(\left|\frac{z_k}{r'}\right|^{n}\right)\right) \tag{3.24}$$

provided that $\Delta_n(z) \neq 0$. From (3.21) and (3.24) we find the monic form

$$\hat{\phi}_n(z) = [\omega(z)]^2 + O\left(\left|\frac{z_k}{r'}\right|^n\right). \tag{3.25}$$

For the case of distinct poles, we let $r' \to r$ and then the results (3.15) and (3.16) follow from (3.25). Now we consider cases of confluence. If any of the points z_i in (3.8) or (3.12) are repeated, so that $\mathbf{f}(z)$ has a multipole at z_i, we may re-express $\omega(z)$ without loss of generality as

$$\omega(z) = \prod_{j=1}^{\kappa} (z - z_j')^{m_j}, \tag{3.26}$$

where the z_j' are distinct and $\{z_j'\}_{j=1}^{\kappa} = \{z_i\}_{i=1}^{k}$. From (3.10) we see that small changes in the coefficients in the polynomial form of $\omega(z)$ induce small changes in the coefficients $\mathbf{c}_i$ of $\mathbf{f}(z)$. Consider small changes which make the zeros of $\omega(z)$ distinct, say at $z_1, z_2, ..., z_k$, but preserve the property that all $\mathbf{g}(z_i) \cdot \mathbf{g}^*(z_i) \neq 0$. Because $\phi_n(z)$ is a continuous function of its parameters $\mathbf{c}_i$ (see (2.24)), the results (3.24) and (3.25) remain true in the confluent limit.

The results (3.13) and (3.14) follow using the original proof [12], and there is no need to repeat it here. ∎

In fact, we can readily obtain a stronger result than is implied by the statement of Theorem 3.1 about the rate of convergence of the denominator polynomials at the poles of $\mathbf{f}(z)$. Again, by expanding the contours in (3.17), we find that

$$\phi_n(z_j') = O\left(\left|\frac{z_j'}{r'}\right|^{2nm_j}\right) \prod_{i=1}^{k} z_i^{-2n},$$

similarly to (3.22). From (3.23) and then by taking the limiting value $r' \to r$, we obtain

$$\limsup_{n \to \infty} |\hat{\phi}_n(z_j')|^{1/n} \leqslant \left|\frac{z_j'}{r}\right|^{2m_j}, \qquad j = 1, 2, ..., k. \tag{3.27}$$

As a corollary to Theorem 3.1, we state a result which is a generalisation of the extension theorem of Graves-Morris and Saff [13]. In contrast to the hypothesis (3.12) of the main theorem, we now consider cases in which

$$\mathbf{g}(z_j) \cdot \mathbf{g}^*(z_j) = 0 \tag{3.28}$$

for some values of j.

COROLLARY 3.1. *The assumption* (3.10) *that* $\mathbf{f}(z)$ *has the representation*

$$\mathbf{f}(z) = \mathbf{g}(z)/\omega(z),$$

where $\omega(z)$ *is the real symmetric polynomial as expressed by* (3.8), (3.9), *and* (3.26), *is continued together with*

$$\mathbf{g}(z'_j) \neq \mathbf{0}, \qquad j = 1, 2, ..., \kappa.$$

If $\mathbf{g}(z'_j) \cdot \mathbf{g}^*(z'_j) = 0$ *for some value of j, let* λ_j *denote the precise order of the zero of* $\mathbf{g}(z) \cdot \mathbf{g}^*(z)$ *at* z'_j, *and use the convention that* $\lambda_j = 0$ *if* $\mathbf{g}(z'_j) \cdot \mathbf{g}^*(z'_j) \neq 0$. *Assume that* $\lambda_j < 2m_j$ *and define*

$$2v := \sum_{j=1}^{\kappa} (2m_j - \lambda_j). \tag{3.29}$$

Let $\mathbf{R}_n(z)$ *be the vector Padé approximant of type* $[n/2v]$ *for* $\mathbf{f}(z)$, *as in* (3.1), *and let* $\phi_n(z)$ *be its associated denominator polynomial, as defined by* (2.23). *Then the convergence results* (3.13)–(3.16) *of Theorem* 3.1 *hold as stated under the more general conditions of* (3.28) *and* (3.29). *Moreover*

$$\limsup_{n \to \infty} |\hat{\phi}_n(z'_j)|^{1/n} \leqslant \left| \frac{z'_j}{r} \right|^{2m_j - \lambda_j}, \qquad j = 1, 2, ..., \kappa. \tag{3.30}$$

Proof. We define $H(z)$, $h(z)$, and $S = \{z'_1, z'_2, ..., z'_k\}$ by

$$H(z) := \frac{\mathbf{g}(z) \cdot \mathbf{g}^*(z)}{\omega(z)^2} = \frac{h(z)}{\prod_{j=1}^{\kappa} (z - z'_j)^{2m_j - \lambda_j}}, \tag{3.31}$$

where $h(z)$ is analytic in $|z| < r$ and all $h(z'_j) \neq 0$. We call λ_j the nullity of the zero of $\mathbf{g}(z'_j) \cdot \mathbf{g}^*(z'_j)$ at z'_j.

If $\zeta \in S \cap \mathbb{R}$, then $\mathbf{g}(\zeta) \cdot \mathbf{g}^*(\zeta) = |\mathbf{g}(\zeta)|^2 \neq 0$ and there is an index i for which $\zeta = z'_i$ is a pole of $H(z)$ of even multiplicity $2m_i$ and nullity $\lambda_i = 0$.

If $\zeta \in S \cap \{\mathbb{C} \setminus \mathbb{R}\}$, we call ζ a (strictly) complex pole and there exist different indices i, j for which $\zeta = z'_i$ and $\zeta^* = z'_j$. Both $h(z)$ and $H(z)$ are real symmetric, and so $m_i = m_j$, $\lambda_i = \lambda_j$, and z'_i, z'_j are both poles of $H(z)$ of multiplicity $2m_i - \lambda_i$.

We have just seen that the real poles of $H(z)$ are of even multiplicities, and that the complex poles of $H(z)$ occur in complex conjugate pairs. Recalling (3.29), we see that $2v$ is the even integer denoting the number of poles, counting multiplicities, of $H(z)$ in $|z| < r$. Therefore we may consistently enumerate these poles, allowing repetition, as

$$z_1, z_1^*, z_2, z_2^*, ..., z_v, z_v^*.$$

The denominator polynomial $\phi_n(z)$ of degree 2ν is defined by (2.23) with ν replacing k. The proof now follows the proof of the main theorem, with ν replacing k throughout. ▮

The special case of this corollary in which each $\lambda_j = m_j$ was first proved [13] using the diagonally signed matrix theorem. In terms of the statement of Corollary 3.1, the conjecture of Graves-Morris and Saff [12] is that the convergence results (3.13)–(3.16) continue to hold for vector Padé approximants $\mathbf{R}_n(z)$ of type $[n/2\nu]$, where

$$2\nu := \sum_{j=1}^{k} \max\{(2m_j - \lambda_j), 1\}$$

instead of (3.29). Corollary 3.1 contains the assumption that $\lambda_j < 2m_j$, and in this respect the conjecture is only partially proved. However, the proof of Corollary 3.1 depends on the existence of poles of total multiplicity $\sum_{j=1}^{k} (2m_j - \lambda_j)$ in $|z| < r$ and so it seems that Corollary 3.1 probably has the correct generality with respect to the value of 2ν. The results (3.27) and (3.30) are stronger results about the rate of convergence of the denominators at zeros of $\omega(z)$ than is directly implied by (3.15) and (3.16) in the statement of the main theorem. They parallel and strengthen the key property previously found by Graves-Morris and Saff [12, Eq. (3.30)] and used to prove the main theorem. The proofs of Theorem 3.1 and Corollary 3.1 given here are noticeably shorter than the original proofs, and they facilitate the following developments.

The first development is a row convergence theorem for hybrid approximants associated with the real-valued denominator polynomials which are derived from formulations using generalised inverses.

THEOREM 3.2. *A vector-valued function* $\mathbf{f}(z)$ *is given in the form of a MacLaurin series* (2.1) *and also by* (3.10) *in which it is assumed that* $\omega(z)$ *is real symmetric and that*

$$\mathbf{g}(z_i) \cdot \mathbf{g}^*(z_i) \neq 0, \qquad i = 1, 2, ..., k.$$

Let $\boldsymbol{\rho}_n(z)$ *be the vector-valued Padé approximant of type* $[n/k]$ *for* $\mathbf{f}(z)$, *as in* (3.3), *and let* $\sigma_n(z)$ *be its associated denominator polynomial, as defined by* (2.26). *Then*

$$\lim_{n \to \infty} \boldsymbol{\rho}_n(z) = \mathbf{f}(z), \qquad z \in D_r^-$$

and the rate of convergence is governed by

$$\limsup_{n\to\infty} \|\mathbf{f}-\boldsymbol{\rho}_n\|_K^{1/n} \leqslant \mu/r,$$

where K is any compact subset of $D_r^- \cap \{z\in\mathbb{C} : |z| \leqslant \mu\}$ for any $\mu < r$.

The (monic) denominator polynomials converge as

$$\lim_{n\to\infty} \hat{\sigma}_n(z) = \omega(z)$$

and their rate of convergence is governed by

$$\lim_{n\to\infty} \|\hat{\sigma}_n - \omega\|_E^{1/n} \leqslant |z_k|/r,$$

where E is any compact subset of $\mathbb{C}$.

Proof. The proof follows that of Theorem 3.1 closely. Notice that (3.17) is replaced by

$$\sigma_n(z) = \frac{1}{k!}\iint\cdots\int \prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)(x_i - y_j)(y_i - x_j)$$
$$\times \prod_{j=1}^{k} (z - x_j)\, \frac{x_j^{-n-1} y_j^{-n-1}}{(2\pi \mathrm{i})^2}\, \frac{\mathbf{g}(x_j) * \mathbf{g}(y_j)}{\omega(x_j)\,\omega(y_j)}\, dx_j\, dy_j, \qquad (3.32)$$

that (3.18) is replaced by

$$\Delta_n(z) := \frac{1}{k!} \sum_{l_1=1}^{k} \cdots \sum_{l_k=1}^{k} \sum_{l_1'=1}^{k} \cdots \sum_{l_k'=1}^{k} \prod_{i<j} (z_{l_i} - z_{l_j})(z_{l_i'} - z_{l_j'})(z_{l_i} - z_{l_j'})(z_{l_i'} - z_{l_j})$$
$$\times \prod_{j=1}^{k} (z - z_{l_j})\, z_{l_j}^{-n-1} z_{l_j'}^{-n-1}\, \frac{\mathbf{g}(z_{l_j}) \cdot \mathbf{g}^*(z_{l_j'})}{\omega'(z_{l_j})\,\omega'(z_{l_j'})},$$

and that (3.21) is replaced by

$$\hat{\Delta}_n(z) = \omega(z). \qquad \blacksquare$$

The result corresponding to (3.27) is

$$\limsup_{n\to\infty} |\hat{\sigma}_n(z_j')|^{1/n} \leqslant \left|\frac{z_j'}{r}\right|^{m_j}.$$

The statement of Corollary 3.1 applies identically as a corollary to Theorem 3.2, except that (3.30) is replaced by

$$\limsup_{n\to\infty} |\hat{\sigma}_n(z'_j)|^{1/n} \leqslant \left|\frac{z'_j}{r}\right|^{m_j-\lambda_j}, \qquad j=1, 2, ..., \kappa,$$

and its proof is almost identical too.

Next, we state Roberts' row convergence theorem for approximants formed using the vector inverses of a complex Clifford algebra, and then the corresponding theorem for the hybrid approximants. It is interesting to contrast his hypotheses that $\mathbf{g}(z_i)\cdot\mathbf{g}(z_i)\neq 0$ with (3.12) and to notice the fact that his theorems do not require $\omega(z)$ to be real symmetric. Roberts used Saff's 1972 method to prove results which, shorn of their Clifford elements, are equivalent to the following theorem:

THEOREM 3.3 (Roberts [22]). *A vector-valued function* $\mathbf{f}(z)$ *is given in the form of a MacLaurin series* (2.1) *and also by* (3.10) *and it is assumed that*

$$\mathbf{g}(z_i)\cdot\mathbf{g}(z_i)\neq 0, \qquad i=1, 2, ..., k.$$

Let $\mathbf{R}_n(z)$ *be the vector-valued Padé approximant of type* $[n/2k]$ *for* $\mathbf{f}(z)$, *as in* (3.5), *and let* $\phi_n^{\mathrm{C}}(z)$ *be its associated denominator polynomial, as defined by* (2.23). *Then*

$$\lim_{n\to\infty} \mathbf{R}_n^{\mathrm{C}}(z)=\mathbf{f}(z), \qquad z\in D_r^-$$

and the rate of convergence is governed by

$$\limsup_{n\to\infty} \|\mathbf{f}-\mathbf{R}_n^{\mathrm{C}}\|_K^{1/n}\leqslant \mu/r,$$

where K *is any compact subset of* $D_r^-\cap\{z\in\mathbb{C}: |z|\leqslant\mu\}$ *for any* $\mu<r$.

The (*monic*) *denominator polynomials converge as*

$$\lim_{n\to\infty} \hat{\phi}_n^{\mathrm{C}}(z)=(\omega(z))^2$$

and their rate of convergence is governed by

$$\limsup_{n\to\infty} \|\hat{\phi}_n^{\mathrm{C}}-\omega^2\|_E^{1/n}\leqslant |z_k|/r,$$

where E *is any compact subset of* $\mathbb{C}$.

Proof. The proof is almost identical to that of Theorem 3.1, but notice, for example, that (3.17) is replaced by

$$\phi_n^{\mathrm{C}}(z) = \frac{1}{k!} \iint \cdots \int \prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)(x_i - y_j)(y_i - x_j)$$
$$\times \prod_{j=1}^{k} (z - x_j)(z - y_j) \frac{x_j^{-n-1} y_j^{-n-1}}{(2\pi \mathrm{i})^2} \frac{\mathbf{g}(x_j) \cdot \mathbf{g}(y_j)}{\omega(x_j)\, \omega(y_j)} \, dx_j \, dy_j. \quad \blacksquare$$

THEOREM 3.4. *A vector-valued function* $\mathbf{f}(z)$ *is given in the form of a MacLaurin series* (2.1) *and also by* (3.10) *and it is assumed that*

$$\mathbf{g}(z_i) \cdot \mathbf{g}(z_i) \neq 0, \qquad i = 1, 2, ..., k.$$

Let $\boldsymbol{\rho}_n^{\mathrm{C}}(z)$ *be the vector-valued Padé approximant of type* $[n/k]$ *for* $\mathbf{f}(z)$, *as in* (3.1), *and let* $\sigma_n^{\mathrm{C}}(z)$ *be its associated denominator polynomial, as defined by* (2.23). *Then*

$$\lim_{n \to \infty} \boldsymbol{\rho}_n^{\mathrm{C}}(z) = \mathbf{f}(z), \qquad z \in D_r^-$$

and the rate of convergence is governed by

$$\limsup_{n \to \infty} \|\mathbf{f} - \boldsymbol{\rho}_n^{\mathrm{C}}\|_K^{1/n} \leqslant \mu/r,$$

where K is any compact subset of $D_r^- \cap \{z \in \mathbb{C} : |z| \leqslant \mu\}$ *for any* $\mu < r$.

The (monic) denominator polynomials converge as

$$\lim_{n \to \infty} \hat{\sigma}_n^{\mathrm{C}}(z) = \omega(z)$$

and their rate of convergence is governed by

$$\limsup_{n \to \infty} \|\hat{\sigma}_n^{\mathrm{C}} - \omega\|_E^{1/n} \leqslant |z_k|/r,$$

where E is any compact subset of $\mathbb{C}$.

Proof. The proof is virtually identical to that of Theorem 3.3, but notice that (3.32) is replaced by

$$\sigma_n^{\mathrm{C}}(z) = \frac{1}{k!} \iint \cdots \int \prod_{i<j}^{k} (x_i - x_j)(y_i - y_j)(x_i - y_j)(y_i - x_j)$$
$$\times \prod_{j=1}^{k} (z - x_j) \frac{x_j^{-n-1} y_j^{-n-1}}{(2\pi \mathrm{i})^2} \frac{\mathbf{g}(x_j) \cdot \mathbf{g}(y_j)}{\omega(x_j)\, \omega(y_j)} \, dx_j \, dy_j.$$

4. CONCLUSION

We have compared and contrasted four different types of vector Padé approximants, each with its distinctive merits. The type of the approximant is determined by two factors, namely (i) whether the coefficients of the denominator are necessarily real or if they can be complex valued, and (ii) whether the denominator has degree (called $2k$) which is double that of the equivalent Padé approximant or if the denominator is of the hybrid variety (of degree k). We have shown that all four types admit row convergence theorems analogous to de Montessus' 1902 theorem.

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